

A SMOOTH VARIATION OF BAAS-SULLIVAN THEORY AND POSITIVE SCALAR CURVATURE

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ABSTRACT. Let M be a smooth closed spin (resp. oriented and totally non-spin) manifold of dimension $n \geq 5$ with fundamental group π . It is stated, e.g. in [RS01], that M admits a metric of positive scalar curvature (pscm) if its orientation class in $ko_n(B\pi)$ (resp. $H_n(B\pi; \mathbb{Z})$) lies in the subgroup consisting of elements which contain pscm representatives. This is 2-locally verified loc. cit. and in [Sto94]. After inverting 2 it was announced that a proof would be carried out in [Jun], but this work has never appeared in print. The purpose of our paper is to present a self-contained proof of the statement with 2 inverted.

1. INTRODUCTION

A basic question in Riemannian geometry is whether a given smooth closed manifold M admits a metric of positive scalar curvature or not. Bordism theory is an important tool to approach this problem. On the one hand the surgery lemma (cf. [GL80],[SY79]) guarantees that under mild conditions the existence of a pscm is invariant under bordism. On the other hand, in case M admits a spin structure, a certain characteristic class $\alpha(M)$, again invariant under bordism, grants an obstruction to the existence of a pscm (cf. [Lic63],[Hit74]).

Let X be a space and $G = Spin$ or SO . As usual we denote the bordism groups of spin resp. oriented manifolds in X by $\Omega_*^G(X)$. An element $[M, f] \in \Omega_n^G(X)$ is a bordism class of continuous maps $f: M \rightarrow X$ where M is a smooth closed spin resp. oriented manifold of dimension n . We set

$${}^+\Omega_n^G(X) := \{[M, f] \in \Omega_n^G(X) \mid M \text{ admits a pscm}\}.$$

In $\Omega_n^G(X)$ addition is given by the disjoint union of manifolds and taking inverses by reversing the spin structure resp. orientation. Hence ${}^+\Omega_n^G(X)$ in fact becomes a subgroup of $\Omega_n^G(X)$. One can combine the surgery lemma with methods from the proof of the s-cobordism theorem to obtain the following existence result:

Theorem 1.1 ([GL80],[RS95]). *Let M be a smooth connected closed manifold of dimension $n \geq 5$ with fundamental group π . Furthermore, let $B\pi$ be the classifying space of π and $f: M \rightarrow B\pi$ the classifying map of the universal covering of M . Then the following holds:*

- (1) *If M admits a spin structure it carries a pscm if and only if $[M, f] \in {}^+\Omega_n^{Spin}(B\pi)$.*

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- (2) If M is orientable and totally non-spin, i.e. its universal cover does not admit a spin structure, it carries a pscm if and only if $[M, f] \in {}^+\Omega_n^{SO}(B\pi)$.

It is desirable to pass from the bordism groups of $B\pi$ to simpler groups which are easier to compute. In the oriented case we have the well-known map

$$U: \Omega_n^{SO}(X) \rightarrow H_n(X; \mathbb{Z})$$

which sends an element $[M, f]$ to the image of the fundamental class $[M] \in H_n(M; \mathbb{Z})$ under the induced map of f in homology. Recall that in stable homotopy theory spectra determine homology theories and vice versa. The corresponding map in the spin case is the Atiyah-Bott-Shapiro orientation [ABS67]

$$A: \Omega_n^{Spin}(X) \rightarrow ko_n(X)$$

where $ko_n(-)$ denotes the homology theory associated to the connective cover of the real K -theory spectrum KO . We set $ko_n^+(X) = A({}^+\Omega_n^{Spin}(X))$ and $H_n^+(X) = U({}^+\Omega_n^{SO}(X))$. Theorem 4.11 in [RS01] states:

Theorem 1.2. *Under the assumptions of Thm. 1.1 the following holds:*

- (1) *If M admits a spin structure it carries a pscm if and only if $A([M, f]) \in ko_n^+(B\pi)$.*
- (2) *If M is orientable and totally non-spin it carries a pscm if and only if $U([M, f]) \in H_n^+(B\pi)$.*

To prove this statement one has to show

Theorem 1.3. $\ker A \subset {}^+\Omega_*^{Spin}(B\pi)$ and $\ker U \subset {}^+\Omega_*^{SO}(B\pi)$.

Such an inclusion of Abelian groups can be shown by proving it *localized* at 2, i.e. after tensoring with $\mathbb{Z}_{(2)} := \{\frac{a}{b} \in \mathbb{Q} \mid b \text{ prime to } 2\}$ and after *inverting* 2, i.e. after tensoring with $\mathbb{Z}[\frac{1}{2}]$.

- (1) $\ker A \otimes \mathbb{Z}_{(2)} \subset {}^+\Omega_*^{Spin}(B\pi) \otimes \mathbb{Z}_{(2)}$ is proved by Stolz [Sto94] using splitting results of $MSpin$ -module spectra.
- (2) $\ker U \otimes \mathbb{Z}_{(2)} \subset {}^+\Omega_*^{SO}(B\pi) \otimes \mathbb{Z}_{(2)}$ can be deduced from the Atiyah-Hirzebruch spectral sequence (sketched in [RS01]).

After inverting 2 it is mentioned in [RS01] that there is a proof by Rainer Jung [Jun], for both the spin and the oriented case, based on the Baas-Sullivan theory of bordism with singularities. To the knowledge of the author, experts in this field agree that Jung's proof is probably correct. However, this proof is not available to the public (and in fact unknown to us). Hence one cannot verify its details, it is unclear how much technical effort is needed and generalizations or modifications cannot be carried out. Due to these reasons we shall fill this gap in the literature.

The strategy of our proof of Thm. 1.3 with 2 inverted is as follows. Let $MSpin$ resp. MSO denote the spin resp. oriented Thom spectrum and $H\mathbb{Z}$ the integer Eilenberg-MacLane spectrum. The orientation maps A and U are induced by spectrum maps $a: MSpin \rightarrow ko$ and $u: MSO \rightarrow H\mathbb{Z}$. We consider the fibrations

$$(1) \widehat{MSpin} \xrightarrow{i} MSpin \xrightarrow{a} ko,$$

$$(2) \widehat{MSO} \xrightarrow{i} MSO \xrightarrow{u} H\mathbb{Z}$$

where \widehat{MSpin} and \widehat{MSO} denote the homotopy fibers of a and u .

All groups and spectra are considered after inverting 2. One can prove that the kernel of the induced map on coefficients, i.e. homotopy groups, $a_*: MSpin_* \rightarrow ko_*$ (resp. $u_*: MSO_* \rightarrow H\mathbb{Z}_*$) is generated by pscm manifolds, cf. [KS93, Sec. 4] (resp. [GL80]). By means of these sets of pscm generators we shall give a geometric interpretation of the homology theories associated to \widehat{MSpin} and \widehat{MSO} in terms of smooth manifolds. It turns out that these manifolds also carry a pscm. Since $\ker A = \text{im}(I: \widehat{MSpin}_*(X) \rightarrow MSpin_*(X))$ (and analogous in the oriented case) this proves Thm. 1.3 with 2 inverted.

Overview. Let \mathcal{P} be a family of smooth closed manifolds. In section 2 of our paper we shall introduce a homology theory $\mathcal{P}_*(-)$, which we call the *bordism spanned by \mathcal{P}* . It is related to Baas-Sullivan theory as follows. Based on ideas of Sullivan [Sul67] Baas [Baa73] introduced homology theories which provide a geometric description of singular homology by means of manifolds with singularities. Removing neighborhoods of these singularities Botvinnik [Bot92, Ch. 1] obtains a description by manifolds with additional structures on their boundaries. It is said in [Bot92] that these boundaries themselves lead to a homology theory, loc. cit. denoted by $MG_*^{\Sigma\Gamma(1)}(-)$.

We cannot see obvious smooth structures on the manifolds which are used in the construction of $MG_*^{\Sigma\Gamma(1)}(-)$. The theory $\mathcal{P}_*(-)$ shall be a smooth variation of $MG_*^{\Sigma\Gamma(1)}(-)$. Elements in $\mathcal{P}_*(-)$ are represented by *smooth* manifolds with additional structure. We directly verify that $\mathcal{P}_*(-)$ satisfies the Eilenberg-Steenrod axioms. Afterwards we compute the coefficients \mathcal{P}_* in our cases of interest and show how this leads to the description of the homotopy fibers \widehat{MSpin} and \widehat{MSO} .

In section 3 we shall prove our geometric result that the manifolds used in our description of $\mathcal{P}_*(-)$ carry a pscm. With respect to constructing pscm, our description, which is based on smooth manifolds right away, seems to be more convenient than classical Baas-Sullivan theory and its further development by Botvinnik.

We note that our treatment of $\mathcal{P}_*(-)$ is self-contained and can be considered as an alternative approach to Baas-Sullivan theory.

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2. A SMOOTH VARIATION OF BAAS-SULLIVAN THEORY

We shall start with some preliminary remarks. Let $\mathbb{H}_i^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$. As usual a smooth n -dimensional manifold M with boundary is modelled on \mathbb{H}_n^n .

We call subsets N_1, \dots, N_k of M a *transversal family of submanifolds* if for all $1 \leq i_1 < \dots < i_l \leq k$ around every point in $N_{i_1} \cap \dots \cap N_{i_l}$ there exists a chart $\psi: U \rightarrow \mathbb{H}_n^n$ of M and an injective map $s: \{1, \dots, l\} \rightarrow \{1, \dots, n-1\}$ such that

$$\psi(U \cap N_{i_j}) = \psi(U) \cap \mathbb{H}_{s(j)}^n$$

simultaneously for all $1 \leq j \leq l$.

Let M and N denote smooth manifolds and let $A \subset M$ be a subset. A map $f: A \rightarrow N$ is called *smooth* if f is the restriction of a smooth map $M \rightarrow N$.

Finally, all upcoming manifolds are supposed to be oriented (resp. equipped with a spin structure) and we assume that all diffeomorphisms between manifolds preserve the orientation (resp. spin structure).

Now let $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ be a finite family of smooth closed manifolds. For $I \subset \{1, \dots, k\}$ we set $P_I := \prod_{i \in I} P_i$.

Definition 2.1. An n -dimensional \mathcal{P} -manifold is a tuple $(M, (A_i)_{1 \leq i \leq k}, (B_I, \phi_I)_{I \subset \{1, \dots, k\}})$ such that

- M is a smooth n -dimensional manifold.
- A_1, \dots, A_k is a transversal family of smooth n -dimensional submanifolds, closed as subsets, and the interiors of A_i cover M .
- For all $I \subset \{1, \dots, k\}$, B_I is a subset of some smooth manifold C_I and ϕ_I is a map $A_I := \cap_{i \in I} A_i \rightarrow P_I \times C_I$ which is a diffeomorphism onto $P_I \times B_I$.
- For $J \subset I$ the map

$$\phi_J \circ \phi_I^{-1}: P_J \times P_{I-J} \times B_I = P_I \times B_I \rightarrow P_J \times B_J$$

is of the form $(x, y) \mapsto (x, \phi_J^I(y))$ where $x \in P_J$, $y \in P_{I-J} \times B_I$ and $\phi_J^I: P_{I-J} \times B_I \hookrightarrow B_J$ is some map.

We agree that \mathbb{H}_n^n always denotes the model space of the surrounding manifold M . Let us call $A_i \subset M$ the P_i -part of a \mathcal{P} -manifold M . If all but one B_i are empty we call M a P_i -manifold.

Definition 2.2. Let X be a space and $A \subset X$. An n -dimensional \mathcal{P} -manifold in (X, A) is a tuple $(M, f, (A_i)_{1 \leq i \leq k}, (B_I, \phi_I)_{I \subset \{1, \dots, k\}}, (f_i)_{1 \leq i \leq k})$ such that

- $(M, (A_i)_{1 \leq i \leq k}, (B_I, \phi_I)_{I \subset \{1, \dots, k\}})$ is a compact n -dimensional \mathcal{P} -manifold.
- $f: (M, \partial M) \rightarrow (X, A)$ and $f_i: B_i \rightarrow X$ are continuous maps such that the diagram

$$\begin{array}{ccc} A_i & \xrightarrow{f} & X \\ \downarrow \phi_i & & \uparrow f_i \\ P_i \times B_i & \xrightarrow{pr} & B_i. \end{array}$$

commutes for all i .

In the sequel we fix a family $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ of smooth closed manifolds and write $(M, f, A_i, B_I, \phi_I, f_i)$, (M, f, A_i) or (M, f) short for a \mathcal{P} -manifold in (X, A) .

Definition 2.3. An n -dimensional \mathcal{P} -manifold $(M, F, A_i, B_I, \phi_I, f_i)$ in (X, A) is said to \mathcal{P} -bord if there exists a tuple $(\hat{M}, \hat{f}, \hat{A}, \hat{B}_I, \hat{\phi}_I, \hat{f}_i)$ such that

- $(\hat{M}, \hat{A}_i, \hat{B}_I, \hat{\phi}_I)$ is a compact $(n+1)$ -dimensional \mathcal{P} -manifold.

- $M \subset \partial \hat{M}$ and for all i one has $\hat{A}_i \cap M = A_i$. In addition, each

$$\hat{\phi}_i \circ \phi_i^{-1}: P_i \times B_i \rightarrow P_i \times \hat{B}_i$$

is of the form $(x, y) \mapsto (x, \omega_i(y))$ for some map ω_i .

- $\hat{F}: \hat{M} \rightarrow X$ and $\hat{f}_i: \hat{B}_i \rightarrow X$ are continuous maps such that $\hat{F}(\partial \hat{M} - M) \subset A$, $\hat{f}|_M = f$ and the diagram

$$\begin{array}{ccc} \hat{A}_i & \xrightarrow{\hat{F}} & X \\ \downarrow \hat{\phi}_i & & \uparrow \hat{f}_i \\ P_i \times \hat{B}_i & \xrightarrow{pr} & \hat{B}_i. \end{array}$$

commutes for all i .

One continues as in ordinary bordism homology. The disjoint union of two \mathcal{P} -manifolds in (X, A) is again a \mathcal{P} -manifold. We say that two n -dimensional \mathcal{P} -manifolds (M, f) and (N, g) in (X, A) are \mathcal{P} -bordant if $(M, f) \cup (-N, g)$ \mathcal{P} -bords.

Lemma 2.4. *\mathcal{P} -bordism defines an equivalence relation.*

Proof. Let $(M, f, A_i, B_I, \phi_I, f_i)$ be \mathcal{P} -manifold in (X, A) . For the proof of reflexivity we consider

$$(2.1) \quad (M \times [0, 1], f \circ pr, A_i \times [0, 1], B_I \times [0, 1], \phi_I \circ pr, f_i \circ pr).$$

By 'straightening the angle' [Con79, p. 8], $M \times [0, 1]$ can be given a differentiable structure. By doing so, $A_1 \times [0, 1], \dots, A_k \times [0, 1]$ becomes a transversal family of submanifolds. There is an induced straightening of B_I for all I and thus 2.1 becomes a \mathcal{P} -bordism between (M, f) and itself.

The symmetry relation is obvious.

To prove transitivity let (M, f) and (N, g) resp. (N, g) and (O, h) be \mathcal{P} -bordant n -dimensional \mathcal{P} -manifolds in (X, A) , say via $(V, F, A_i, B_I, \phi_I, f_i)$ resp. $(W, G, C_i, D_I, \psi_I, g_i)$. Because of transversality one finds charts of V around $A_I \cap \partial V$ and of W around $C_I \cap \partial W$ in which the respective inner boundaries $\overline{\partial A_i - \partial V}$ and $\overline{\partial C_i - \partial W}$ of the P_i -parts lie on a common $\partial \mathbb{H}_j^{n+1}$ for some $j \leq n$ depending on i . Hence, for all i we can glue A_i and C_i along $(A_i \cap \partial V)|_N \cong (C_i \cap \partial W)|_N$ such that $A_1 \cup C_1, \dots, A_k \cup C_k$ becomes a transversal family of submanifolds of the resulting smooth manifold $V \cup W$. Let the P_i -part of N be diffeomorphic to $P_i \times E_i$. By means of point two of Def. 2.3 one recovers E_i as a submanifold of B_i and D_i . Thus, for all i we can also glue B_i and D_i along E_i . One obtains an induces bonding of B_I and D_I for all I . Now the desired \mathcal{P} -bordism between (M, f) and (O, h) is given by

$$(V \cup W, F \cup G, A_i \cup C_i, B_I \cup D_I, \phi_I \cup \psi_I, f_i \cup g_i).$$

□

Denote by $\mathcal{P}_n(X, A)$ the set of all \mathcal{P} -bordism classes of n -dimensional \mathcal{P} -manifolds in (X, A) . Via disjoint union it becomes an Abelian group with zero element the \mathcal{P} -bordism class which \mathcal{P} -bords. A map $g: (X, A) \rightarrow (Y, B)$ induces a group homomorphism $\mathcal{P}_n(X, A) \rightarrow \mathcal{P}_n(Y, B)$ by $[(M, f)] \mapsto [(M, g \circ f)]$. If (M, f) is a \mathcal{P} -manifold in (X, A) then the boundary of M becomes a \mathcal{P} -manifold in A by restriction. It is denoted by $\partial(M, f)$. Then we have an induced map $\partial: \mathcal{P}_n(X, A) \rightarrow \mathcal{P}_{n-1}(A, \emptyset)$ defined by $[(M, f)] \mapsto [\partial(M, f)]$.

Proposition 2.5. *The bordism spanned by \mathcal{P}*

$$\mathcal{P}_*(X, A) := \bigoplus_{n \geq 0} \mathcal{P}_n(X, A)$$

is a homology theory.

Proof. We have to show that $\mathcal{P}_*(X, A)$ satisfies the Eilenberg-Steenrod axioms. One proceeds in the same way as in the case of ordinary bordism homology [Con79, p. 11-13] and additionally takes the local product structures into account. However, the proof of excision requires special attention. For the sake of completeness we shall verify all axioms.

Let $i: A \hookrightarrow X$ and $j: (X, \emptyset) \hookrightarrow (X, A)$ denote the inclusions. Obviously $\mathcal{P}_*(-)$ is a functor from the category of pairs of topological spaces (with continuous maps as morphisms) to the category of Abelian groups. It remains to show:

Homotopy axiom. Let $g, h: (X, A) \rightarrow (Y, B)$ be homotopic via $H: (X, A) \times [0, 1] \rightarrow (Y, B)$. Then $g_* = h_*: \mathcal{P}_n(X, A) \rightarrow \mathcal{P}_n(Y, B)$.

Let $(M, f, A_i, B_I, \phi_I, f_i)$ be a \mathcal{P} -manifold in (X, A) . We define

$$G: M \times [0, 1] \rightarrow Y, (x, t) \mapsto H(f(x), t).$$

By straightening the angle $M \times [0, 1]$ can be equipped with the structure of a \mathcal{P} -manifold. Then

$$(M \times [0, 1], G, A_i \times [0, 1], B_I \times [0, 1], \phi_I \times \text{id}, H \circ (f_i \times \text{id}))$$

becomes a \mathcal{P} -bordism between (M, gf) and (M, hf) .

Exactness axiom. The sequence

$$\dots \xrightarrow{\partial} \mathcal{P}_n(A) \xrightarrow{i_*} \mathcal{P}_n(X) \xrightarrow{j_*} \mathcal{P}_n(X, A) \xrightarrow{\partial} \mathcal{P}_{n-1}(A) \xrightarrow{i_*} \dots$$

is exact.

It is clear that $\partial j_* = 0$ and $i_* \partial = 0$. Let $[(M, f)] \in \mathcal{P}_n(A)$, a zero \mathcal{P} -bordism for $(M, j i f)$ is given by $(M \times [0, 1], j i f pr)$, hence $j_* i_* = 0$.

Let $[(M, f, A_i, B_I, \phi_I, f_i)] \in \mathcal{P}_n(X, A)$ be in the kernel of ∂ . Then $\partial(M, f)$ bords in A , i.e. there exists a zero \mathcal{P} -bordism $(W, g, C_i, D_I, \psi_I, g_i)$ for $\partial(M, F)$ in A . As in the proof of transitivity in Lemma 2.4 we can glue A_i and C_i along $A_i \cap \partial M \cong C_i \cap \partial W$ for all i to obtain a closed \mathcal{P} -manifold N and a map $(f \cup g): N \rightarrow X$. Now $(N \times [0, 1], (f \cup g) \circ pr)$ is a \mathcal{P} -bordism between $(N, j \circ (f \cup g))$ and (M, f) in (X, A) . The cases $\ker j_* \subset \text{im } i_*$ and $\ker i_* \subset \text{im } \partial$ are obvious.

Excision axiom. Let U be an open subset of X such that $\overline{U} \subset \overset{\circ}{A}$, then the inclusion $i: (X - U, A - U) \hookrightarrow (X, A)$ induces an isomorphism

$$i_*: \mathcal{P}_n(X - U, A - U) \xrightarrow{\cong} \mathcal{P}_n(X, A).$$

First we show that i_* is epic: Let $(M, f, A_i, B_I, \phi_I, f_i)$ be a \mathcal{P} -manifold in (X, A) . We are looking for a smooth submanifold $N \subset M$ such that $f^{-1}(X - \mathring{A}) \subset N$ and $f^{-1}(\overline{U}) \cap N = \emptyset$. In addition, N shall respect the local product structures in the sense that $\phi_I(A_I \cap N) = P_I \times C_I$ for some $C_I \subset B_I$. Then it follows that N inherits a \mathcal{P} -structure of M . Now $(N, F|_N)$ defines an element in $\mathcal{P}_n(X - U, A - U)$ and

$$(M \times [0, 1], f \circ pr, A_i \times [0, 1], B_I \times [0, 1], \phi_I \times \text{id}, f_i \circ pr)$$

is a \mathcal{P} -bordism between $(N, i \circ f|_N)$ and (M, f) in (X, A) .

The construction of N requires a preliminary observation. Until the end of this proof we shall denote the 'inner' boundary $\overline{\partial A_i} - \partial \overline{M}$ by ∂A_i , likewise for comparable sets. If $\partial A_i \times [0, 1]$ is a collar neighborhood for $\partial A_i \subset A_i$, say $\partial A_i \times \{0\} = \partial A_i$, we set

$$A_i^t := A_i - (\partial A_i \times [0, t))$$

for all $0 \leq t \leq 1$ (see figure 1). Now one observes that there exists collar neighborhoods $\partial A_i \times [0, 1]$ for all i such that for arbitrary sequences of numbers $0 \leq t_1, \dots, t_k \leq 1$ and by suitable restrictions $(M, f, A_i^{t_i})$ become \mathcal{P} -bordant \mathcal{P} -manifolds in (X, A) .

Now let us construct N . We set $Q := f^{-1}(X - \mathring{A})$ and $R := f_I^{-1}(\overline{U})$. One can show [Con79, Lemma 3.1] that there exists an n -dimensional submanifold $N_0 \subset M$, closed as a subset, such that $Q \subset N_0$ and $R \cap N_0 = \emptyset$. Clearly, N_0 does not have to respect the local product structures. Therefore we shall modify N_0 as follows. Set $B'_i := pr_{B_i}(\phi_i(N_0 \cap A_i))$ and consider the *saturation of the P_i -fibers*

$$N_1 := \bigcup_{i=1}^k \left(\bigcup_{b \in B'_i} \phi_i^{-1}(P_i \times \{b\}) \right).$$

Due to the condition $\cap_{i \in I} A_i \cong P_I \times B_I$ this N_1 respects the local product structures. In addition, as f locally factors over B_I one concludes $f(N_0) = f(N_1)$. Now N_1 is the union of manifolds modelled on $\mathbb{H}_1 \cap \mathbb{H}_2$. The non-smooth points of N_1 only occur on $C_1 := \cup_{i=1}^k \partial A_i$. With respect to the metric induced by the collar neighborhoods let U_1 be an open $1/k$ -neighborhood of C_1 . One finds a smooth submanifold $N'_1 \subset M$ such that $N'_1 - U_1 = N_1 - U_1$. In view of continuity N'_1 can be chosen such that any longer $Q \subset N'_1$ and $R \cap N'_1 = \emptyset$. Note that N'_1 respects the local product structures except on U_1 .

Now we replace A_i by $A_i^{1/k}$ for all i and repeat the above procedure. The saturations of the P_i -fibers with respect to the $A_i^{1/k}$'s yield an $N_2 \subset M$. One merely has to perform this saturation step on U_1 and, again, non-smooth points only appear on $\cup_{i=1}^k \partial A_i^{1/k}$. Hence the non-smooth points of N_2 occur on an open $1/k$ -neighborhood U_2 of

$$\begin{aligned} C_2 &:= C_1 \cap \left(\bigcup_{i=1}^k \partial A_i^{1/k} \right) \\ &= \bigcup_{1 \leq i, j \leq k} \left(\partial A_i \cap \partial A_j^{1/k} \right). \end{aligned}$$

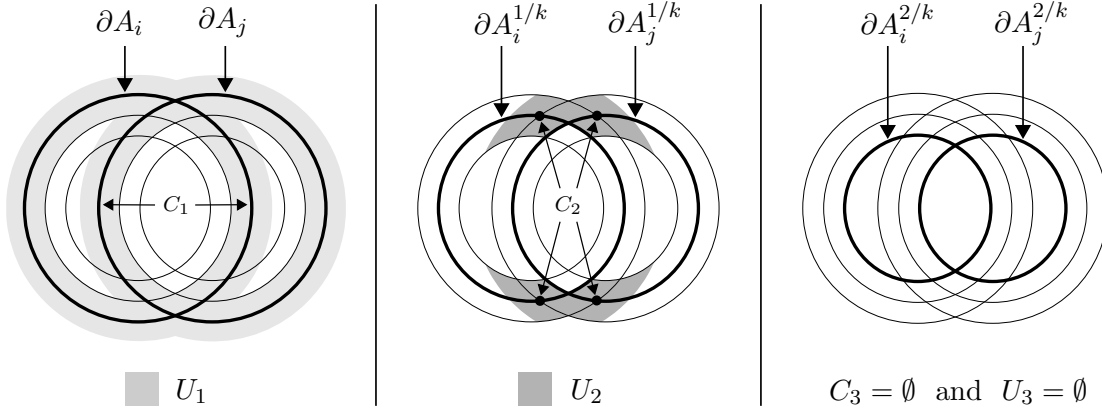


FIGURE 1. Saturation and missing product structures after smoothing

Similar as above there is a smooth submanifold $N'_2 \subset M$ which satisfies $N'_2 - U_2 = N_2 - U_2$ and $Q \subset N'_2$, $R \cap N'_2 = \emptyset$.

Proceeding in this fashion $k - 2$ times more, in each step replacing $A_i^{t/k}$ by $A_i^{(t+1)/k}$, we obtain a smooth submanifold $N'_k \subset M$ which satisfies the local product structures except on an open $1/k$ -neighborhood U_k of

$$C_k := \bigcup_{1 \leq i_1, \dots, i_k \leq k} \left(\partial A_{i_1} \cap \partial A_{i_2}^{1/k} \cap \dots \cap \partial A_{i_k}^{(k-1)/k} \right).$$

Since the interiors of $A_{i_1}, A_{i_2}^{1/k}, \dots, A_{i_k}^{(k-1)/k}$ cover M for all pairwise disjoint $1 \leq i_1, \dots, i_k \leq k$ and $\partial A_{i_j}^{s/k} \cap \partial A_{i_j}^{t/k} = \emptyset$ for $s \neq t$, one concludes that $C_k = \emptyset$ and thus $N := N'_k$ is as desired (see figure 1).

Similarly one sees that i_* is monic: Let (M, f) be an n -dimensional \mathcal{P} -manifold in $(X - U, A - U)$ and $i_*[(M, f)] = 0$. Then there exists a zero \mathcal{P} -bordism (W, g) for $(M, i \circ f)$ in (X, A) . As above we find an $(n + 1)$ -dimensional \mathcal{P} -submanifold $N \subset W$ with $g(N) \cap U = \emptyset$ and $g^{-1}(X - \mathring{A}) \subset N$. It follows that $(N, g|_N)$ is a zero \mathcal{P} -bordism for (M, f) in $(X - U, A - U)$. \square

Now let \mathcal{P} be a, possibly infinite, family of smooth closed manifolds. It is not difficult to extend our definition to that case: Let us denote the family of all finite subsets of \mathcal{P} by \mathfrak{F} . The bordism spanned by $\mathcal{F} \in \mathfrak{F}$ is denoted by $\mathcal{F}_*(-)$. By taking inclusions \mathfrak{F} becomes a directed set and thus we can form the direct limit

$$\mathcal{P}_*(-) := \lim_{\mathcal{F} \in \mathfrak{F}} \mathcal{F}_*(-).$$

As the direct limit preserves exactness $\mathcal{P}_*(-)$ is again a homology theory.

We turn to the computation of the coefficients groups \mathcal{P}_* . Let $G = SO$ or $Spin$. As usual let MG denote the Thom spectrum associated to the oriented resp. spin bordism

theory, i.e. $\Omega_*^G(-) = MG_*(-)$. From now on we fix a *regular sequence* P_1, P_2, \dots of smooth closed manifolds which means that for all $i \geq 1$

$$MG_* / ([P_1], \dots, [P_{i-1}]) \xrightarrow{\times [P_i]} MG_* / ([P_1], \dots, [P_i])$$

is injective, here $([P_1], \dots, [P_{i-1}])$ denotes the ideal generated by $[P_1], \dots, [P_{i-1}]$.

Remark 2.6. One can show cf. [CW11, Prop. 2.7.1.] that in our situation any permutation of P_1, P_2, \dots is again a regular sequence.

Let $\mathcal{P} = \{P_1, P_2, \dots\}$. We have a natural transformation of homology theories $\iota_*: \mathcal{P}_*(-) \rightarrow MG_*(-)$ by forgetting the \mathcal{P} -structure. The following proposition is the crucial step in determining the coefficients.

Proposition 2.7. *ι_* is injective on coefficients.*

Proof. Let \mathfrak{R}_k denote the family of all subsets of \mathcal{P} consisting of k elements. For $\mathcal{R} \in \mathfrak{R}_k$ we have the bordism spanned by \mathcal{R} , denoted by $\mathcal{R}_*(-)$, and the forgetful map $\iota_*^{\mathcal{R}}: \mathcal{R}_*(-) \rightarrow MG_*(-)$. We shall prove the following statement by induction over k from which Prop. 2.7 follows immediately:

Let $\mathcal{R} \in \mathfrak{R}_k$, then $\iota_*^{\mathcal{R}}: \mathcal{R}_* \rightarrow MG_*$ is injective.

$k = 1$: Let $\mathcal{R} \in \mathfrak{R}_1$. In this case a closed \mathcal{R} -manifold M is diffeomorphic to $P_l \times B_l$ for $\{P_l\} = \mathcal{R}$. If $\iota_*^{\mathcal{R}}([M]) = 0$ then $[B_l] = 0$ in MG_* as $[P_l]$ is not a zero divisor. Hence $B_l = \partial W$ for some manifold W . Now the P_l -manifold $P_l \times W$ establishes a zero \mathcal{R} -bordism for M .

$k - 1 \rightarrow k$: Assume that $\iota_*^{\mathcal{S}}: \mathcal{S}_* \rightarrow MG_*$ is injective for all $\mathcal{S} \in \mathfrak{R}_{k-1}$. Let $\mathcal{R} := \{P_{i_1}, \dots, P_{i_k}\} \in \mathfrak{R}_k$ and

$$(M, (A_{i_j})_{1 \leq j \leq k}, (B_I, \phi_I)_{I \subset \{i_1, \dots, i_k\}})$$

a closed \mathcal{R} -manifold. One has to show that $[M] = 0$ in \mathcal{R}_* if $\iota_*^{\mathcal{R}}[M] = 0$. We shall apply, step by step, surgery to the submanifolds $\partial A_{i_j} \subset M$ in order to isolate the P_{i_j} -parts and to obtain a disjoint union of P_{i_j} -manifolds.

It follows from Def. 2.1 that C_{i_j} , together with ϕ_{i_j} , induces a smooth structure on ∂B_{i_j} such that $\phi_{i_j}: \partial A_{i_j} \rightarrow P_{i_j} \times \partial B_{i_j}$ is a diffeomorphism for all $1 \leq j \leq k$. The induction hypothesis becomes applicable by means of

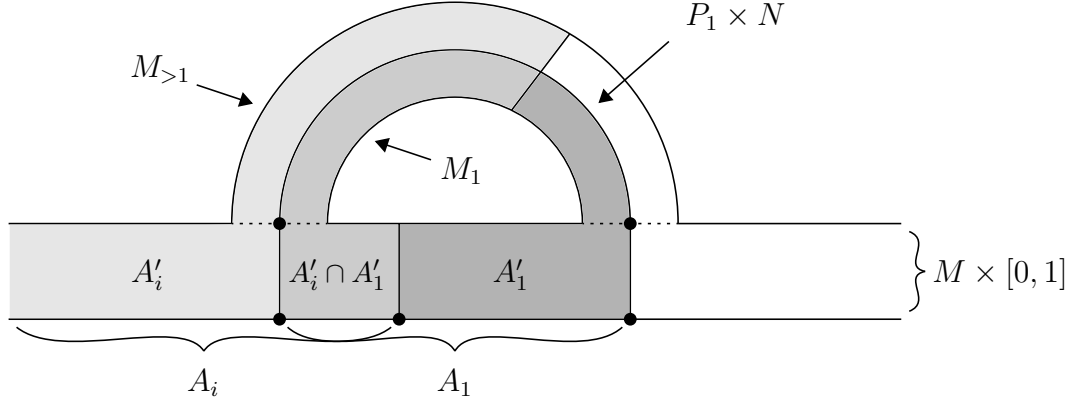
Lemma 2.8. *Let $\mathcal{S} := \mathcal{R} - \{P_{i_j}\}$, i.e. $\mathcal{S} \in \mathfrak{R}_{k-1}$. Then ∂B_{i_j} inherits the structure of an \mathcal{S} -manifold.*

Proof. Let $J \subset \{i_1, \dots, \hat{i}_j, \dots, i_k\}$. There are manifolds $C_J \subset B_{\{i_j\} \cup J}$ such that

$$\phi_{\{i_j\} \cup J}(\partial A_{i_j} \cap A_J) = P_{i_j} \times P_J \times C_J.$$

Then we have an inclusion $\phi_{i_j}^{\{i_j\} \cup J}: P_J \times C_J \hookrightarrow \partial B_{i_j}$ (cf. point four of Def. 2.1) and observe

$$\bigcup_{l \neq j} \phi_{i_j}^{\{i_j, i_l\}}(P_l \times C_l) = \partial B_{i_j}.$$

FIGURE 2. \mathcal{R} -structure of W

Now the induced \mathcal{S} -structure on ∂B_{i_j} is defined by setting

$$\left(\partial B_{i_j}, \left(\phi_{i_j}^{\{i_j, i_l\}} (P_l \times C_l) \right)_{l \neq j}, \left(C_J, \left(\phi_{i_j}^{\{i_j\} \cup J} \right)^{-1} \Big|_{\text{im } \phi_{i_j}^{\{i_j\} \cup J}} \right)_{J \subset \{i_1, \dots, \hat{i}_j, \dots, i_k\}} \right).$$

□

In the first surgery step we consider ∂A_{i_1} . As $[P_{i_1}]$ is not a zero divisor, ∂B_{i_1} is zero bordant in MG_* . Hence, by induction hypothesis, ∂B_{i_1} is zero \mathcal{S} -bordant with $\mathcal{S} := \mathcal{R} - \{i_1\}$, i.e. there exists an \mathcal{S} -manifold (N, C_i) with $\partial N = \partial B_{i_1}$. By abuse of notation, we shall use the indices $\{1, \dots, k\}$ instead of $\{i_1, \dots, i_k\}$ in the sequel. Fix bicollar neighborhoods $\partial A_i \times [-1, 1]$ of $\partial A_i \subset M$, say $\partial A_i \times \{-1\} \subset A_i$, such that $(\partial A_i \times [-1, 1]) \cap A_j$ is a bicollar neighborhood for $(\partial A_i) \cap A_j$ in A_j for all j .

Now we attach $P_1 \times N \times [-1, 1]$ to $M \times [0, 1]$ by identifying

$$(x, y, 1) \in (\partial A_1 \times [-1, 1] \times [0, 1]) \subset (M \times [0, 1]) \text{ with } (\phi_1(x), y) \in (P_1 \times \partial N) \times [-1, 1]$$

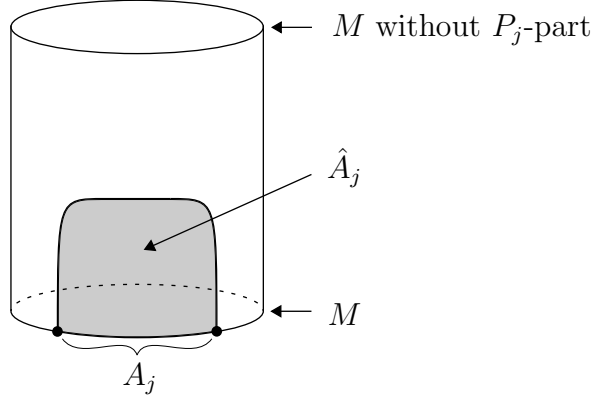
to obtain an oriented (resp. spin) manifold W . It is well-known that the corners of W can be smoothened in a canonical way.

Let us equip W with the structure of an \mathcal{R} -manifold (see figure 2). On $M \times [0, 1]$ we simply set $A'_i := A_i \times [0, 1]$. On $P_1 \times N \times [-1, 1]$ we define $A'_1 := P_1 \times N \times [-1, 0]$ and, for $i > 1$, $A'_i := P_1 \times C_i \times [-1, 1]$. Note that this \mathcal{R} -structure on W induces the given one on $M \times \{0\}$.

By construction W is an \mathcal{R} -bordism between M and the disjoint union of

- $M_1 := A_1 \cup (P_1 \times N)$ glued along $\phi_1(\partial A_1) = \partial(P_1 \times N)$,
- $M_{>1} := (M - \mathring{A}_1) \cup (P_1 \times N)$ glued along $\phi_1(\partial(M - \mathring{A}_1)) = \partial(P_1 \times N)$.

The trace of the bordism induces an \mathcal{R} -structure on $M_{>1}$ with an empty P_1 -part. A priori W induces on M_1 an \mathcal{R} -structure with non-empty P_i -parts, $i > 1$. However, M_1 is

FIGURE 3. Redundant A_j -part

completely covered by the P_1 -part $P_1 \times (B_1 \cup N)$. The following lemma shows that we can ignore redundant subsets A_i , more precisely:

Lemma 2.9. *Let $(M, (A_i)_{1 \leq i \leq k})$ be a closed \mathcal{R} -manifold, $1 \leq j \leq k$ and $M = \cup_{i=1, i \neq j}^k A_i$. Then $(M, (A_i)_{1 \leq i \leq k})$ is \mathcal{R} -bordant to $(M, (A'_i)_{1 \leq i \leq k})$ with $A'_i = A_i$ for $i \neq j$ and $A'_j = \emptyset$.*

Proof. Let A_j be diffeomorphic to $P_j \times B_j$. Choose a collar neighborhood $\partial B_j \times [-1, 0]$, say $\partial B_j \times \{0\} = \partial B_j$, such that for the induced collar neighborhood $\partial A_j \times [-1, 0]$ it is true that $(\partial A_j \times [-1, 0]) \cap A_i$ is a collar neighborhood for $(\partial A_j) \cap A_i$ in A_i for all i . Let $\gamma: [-1, 0] \rightarrow [-1, 0] \times [0, 0.5]$ be a smooth injective convex curve with $\gamma(t) = (t, 0.5)$ for $t < -0.9$ and $\gamma(t) = (0, -t)$ for $t > -0.1$. Now we define the desired \mathcal{R} -bordism $M \times [0, 1] = \cup_{i=1}^k \hat{A}_i$ by (see figure 3) $\hat{A}_i := A_i \times [0, 1]$ for $i \neq j$ and

$$\hat{A}_j := \left\{ (x, s) \in A_j \times [0, 1] \mid s \leq \begin{cases} 0.5 & \text{if } x \in A_j - (\partial A_j \times [-1, 0]) \\ \gamma(t)_2 & \text{if } x = (y, t) \in \partial A_j \times [-1, 0] \\ \max_{\{t \mid \gamma(t)_1 = 0\}} \gamma(t)_2 & \text{if } x \in \partial A_j \times \{0\} \end{cases} \right\}.$$

□

Applying this statement $(k-1)$ -times to M_1 it follows that M_1 becomes a P_1 -manifold. As noted above any permutation of P_1, P_2, \dots, P_k is again a regular sequence. Therefore we may repeat the above surgery procedure applied to the \mathcal{R} -manifold $M_{>1}$. This yields an \mathcal{R} -bordism between $M_{>1}$ on the one hand and a P_2 -manifold M_2 resp. an \mathcal{R} -manifold $M_{>2}$ with empty P_1 - and P_2 -parts on the other. In this fashion we obtain an \mathcal{R} -bordism between M and a disjoint union

$$(2.2) \quad (P_1 \times Q_1) \dot{\cup} \dots \dot{\cup} (P_k \times Q_k),$$

where each $P_i \times Q_i$ is as \mathcal{R} -manifold a P_i -manifold.

To complete the induction step we have to show that if $\iota_*^{\mathcal{R}}[M] = 0$ in MG_* then $[M] = 0$ in \mathcal{R}_* . Note again that any permutation of P_1, \dots, P_k is regular. In MG_* we observe the

following: Since M is zero bordant it follows from 2.2 that for all $1 \leq j \leq k$

$$\sum_i [P_i \times Q_i] = 0 \mod ([P_1], \dots, [\widehat{P_j}], \dots, [P_k]),$$

hence $[Q_j] \in ([P_1], \dots, [\widehat{P_j}], \dots, [P_k])$. Then we conclude that for all $1 \leq s, t \leq k$ there exists a closed manifold Q_{st} such that M is bordant to $\sum_{s,t} P_{\{s,t\}} \times Q_{st}$. Furthermore, for all $1 \leq j, l \leq k$ we look at

$$\sum_{s,t} [P_{\{s,t\}} \times Q_{st}] = 0 \mod ([P_1], \dots, [\widehat{P_j}], \dots, [\widehat{P_l}], \dots, [P_k])$$

and conclude $[Q_{st}] \in ([P_1], \dots, [\widehat{P_j}], \dots, [\widehat{P_l}], \dots, [P_k])$ for $\{s, t\} = \{j, l\}$. Proceeding in this fashion we find a closed manifold Q such that M is bordant to $P_1 \times \dots \times P_k \times Q$.

Now we turn to \mathcal{R}_* . If we remember the P_i -factors in 2.2 it follows from the above observation that there are closed manifolds R_i such that M is \mathcal{R} -bordant to

$$(2.3) \quad (\overline{P_1} \times \dots \times P_k \times R_1) \dot{\cup} \dots \dot{\cup} (P_1 \times \dots \times \overline{P_k} \times R_k),$$

where $\overline{P_1} \times \dots \times P_k \times R_1$ denotes the P_1 -manifold $P_1 \times \dots \times P_k \times R_1$ etc. It is not difficult to see that the specification of the P_i 's in 2.3 is immaterial. In fact, $\overline{P_i} \times P_j$ is \mathcal{R} -bordant to $P_i \times \overline{P_j}$ by means of the \mathcal{R} -bordism $[0, 1] \times P_i \times P_j$ with $[0, \frac{2}{3}] \times P_i \times P_j$ as the P_i -part and $[\frac{1}{3}, 1] \times P_i \times P_j$ as the P_j -part.

We conclude that M is \mathcal{R} -bordant to, say, $\overline{P_1} \times \dots \times P_k \times Q$. Now, since $[P_1 \times \dots \times P_k]$ is not a zero divisor it follows that Q is zero bordant, i.e. $Q = \partial R$ for some R . The P_1 -manifold $\overline{P_1} \times \dots \times P_k \times R$ is the required zero \mathcal{R} -bordism for M . This finishes the induction step. \square

Remark 2.10. Our construction of $\mathcal{P}_*(-)$ and Prop. 2.7 immediately extend to smooth manifolds with other reductions of their structure groups.

Now we show how one obtains the desired description of \widehat{MSO} and \widehat{MSpin} . We consider all spectra and groups after inverting 2. Let us start with the oriented case. By a classical result of Milnor [Mil58] there are closed oriented manifolds Q_1, Q_2, \dots , $\dim Q_i = 4i$, such that

$$(2.4) \quad MSO_* \cong \mathbb{Z}[\frac{1}{2}][[Q_1], [Q_2], \dots]$$

(cf. also [Sto68, p. 180]). For example, we can take complex projective spaces and hypersurfaces of degree $(1, 1)$ in $\mathbb{C}P^n \times \mathbb{C}P^m$ as generators. One concludes that the kernel of $u_*: MSO_* \rightarrow H\mathbb{Z}_*$ coincides with the ideal $([Q_1], [Q_2], \dots)$. We denote the spectrum associated to the bordism spanned by $\{Q_1, Q_2, \dots\}$ by \mathcal{P}^u . In view of 2.4 the sequence Q_1, Q_2, \dots is regular.

Proposition 2.11. *There is a canonical homotopy equivalence $\mathcal{P}^u \simeq \widehat{MSO}$.*

Proof. The natural transformation of homology theories $\iota_*: \mathcal{P}_*^u(_) \rightarrow MSO_*(_)$ corresponds to a spectrum map $\iota: \mathcal{P}^u \rightarrow MSO$. Consider the lifting problem

$$\begin{array}{ccc} & \widehat{MSO} & \\ \nearrow \iota & \downarrow i & \\ \mathcal{P}^u & \xrightarrow{\iota} MSO & \xrightarrow{u} H\mathbb{Z}. \end{array}$$

Since $\widetilde{H}^0(\mathcal{P}^u; \mathbb{Z}[\frac{1}{2}]) = 0$ there exists a unique (up to homotopy) solution $h: \mathcal{P}^u \rightarrow \widehat{MSO}$. On the one hand $i_*: \widehat{MSO}_n \rightarrow MSO_n$ is an isomorphism for all $n > 0$ and $\widehat{MSO}_0 = 0$. On the other hand $\iota_*: \mathcal{P}_n^u \rightarrow MSO_n$ is obviously surjective for all $n > 0$ and, following Lemma 2.7, ι_* is injective. In addition, $\mathcal{P}_0^u = 0$. It follows that $h_*: \mathcal{P}_n^u \rightarrow \widehat{MSO}_n$ is an isomorphism for all $n \geq 0$. All spectra involved are CW -spectra and thus, according to Whitehead's theorem, h is a homotopy equivalence. \square

Now we turn to the spin case. One proceeds like in the oriented case, however, solving the corresponding lifting problem requires some more work. There are closed spin manifolds R_1, R_2, \dots , $\dim R_i = 4i$, such that

$$MSpin_* \cong \mathbb{Z}[\frac{1}{2}][[R_1], [R_2], \dots].$$

As one can show [KS93, Sec. 4], R_1, R_2, \dots can be chosen such that R_i is the total space of a $\mathbb{H}P^2$ -bundle for $i \geq 2$ and the kernel of $a_*: MSpin_* \rightarrow ko_*$ coincides with the ideal $([R_2], [R_3], \dots)$. Similar as above we denote the spectrum associated to the bordism spanned by the regular sequence R_2, R_3, \dots by \mathcal{P}^a .

Proposition 2.12. *There is a canonical homotopy equivalence $\mathcal{P}^a \simeq \widehat{MSpin}$.*

The lifting problem

$$\begin{array}{ccc} & \widehat{MSpin} & \\ \nearrow \iota & \downarrow i & \\ \mathcal{P}^a & \xrightarrow{\iota} MSpin & \xrightarrow{a} ko \end{array}$$

is equivalent to the extension problem

$$\begin{array}{ccc} \mathcal{P}^a & \xrightarrow{\iota} MSpin & \xrightarrow{p} \mathcal{C} \\ & \downarrow a & \nearrow \\ & ko & \end{array}$$

where \mathcal{C} denotes the homotopy cofiber of ι .

Proposition 2.13 (Equiv. to Prop. 2.12). *There is a canonical homotopy equivalence $\mathcal{C} \simeq ko$.*

Remark 2.14. One can show that the homology theory associated to \mathcal{C} admits a description by means of smooth spin manifolds which carry a \mathcal{P}^a -structure on their boundary. Hence we recover a geometric description of $ko_*(-)[\frac{1}{2}]$.

Proof. The idea is that we find an isomorphism on the level of coefficients and then apply the Conner-Floyd theorem to obtain a spectrum map. A similar argument can be found in [Lan76, p. 597]. In order to proceed in this way one has to turn to periodic theories first.

Analogous to the oriented case the image of \mathcal{P}^a under ι_* coincides with the ideal $([R_2], [R_3], \dots)$. Hence we have $\mathcal{C}_* = \mathbb{Z} \left[\frac{1}{2} \right] [p(R_1)]$. Consider now the $MSpin$ -module spectrum $\mathcal{C}[p(R_1)^{-1}]$. As module over $MSpin_*$ one concludes that $\mathcal{C}[p(R_1)^{-1}]_*$ is isomorphic to $\mathbb{Z} \left[\frac{1}{2} \right] [p(R_1), p(R_1)^{-1}]$.

As a generator R_1 of $MSpin_4$ one can use a K3 surface of signature 16. Let $\omega_4 \in ko_4$ denote the image of $[R_1] \in MSpin_4$ under the orientation map $a_*: MSpin_* \rightarrow ko_*$. By Bott periodicity one has $KO_* = \mathbb{Z} \left[\frac{1}{2} \right] [\omega_4, \omega_4^{-1}]$ and hence $ko_* = \mathbb{Z} \left[\frac{1}{2} \right] [\omega_4]$. It follows that there is a unique $MSpin_*$ -module isomorphism $\phi: KO_* \rightarrow \mathcal{C}[p(R_1)^{-1}]_*$.

Now let MSp denote the symplectic Thom spectrum. The real Conner-Floyd theorem [CF66] states that

$$\mu: MSp_*(X) \otimes_{MSp_*} KO_* \rightarrow KO_*(X),$$

induced by the MSp -module structure of KO , is an isomorphism. After inverting 2 one can show that there is a natural equivalence $MSpin \simeq MSp$. Consider now

$$(2.5) \quad \begin{array}{ccc} MSpin_*(X) \otimes_{MSpin_*} KO_* & \xrightarrow{\mu} & KO_*(X) \\ \downarrow \text{id} \otimes \phi & & \\ MSpin_*(X) \otimes_{MSpin_*} \mathcal{C}[p(R_1)^{-1}]_* & \xrightarrow{\eta} & \mathcal{C}[p(R_1)^{-1}]_*(X), \end{array}$$

where η is induced by the $MSpin$ -module structure of $\mathcal{C}[p(R_1)^{-1}]$. It follows that the natural transformation of homology theories

$$\eta \circ (\text{id} \otimes \phi) \circ \mu^{-1}: KO_*(-) \rightarrow \mathcal{C}[p(R_1)^{-1}]_*(-)$$

is an equivalence. Hence there exists a homotopy equivalence $KO \simeq \mathcal{C}[p(R_1)^{-1}]$. The periodization map $ko \rightarrow KO$ induces isomorphisms on non-negative homotopy groups and \mathcal{C} is a connective spectrum. One concludes that the lifting problem

$$\begin{array}{ccc} & & ko \\ & \nearrow \text{dashed} & \downarrow \\ \mathcal{C} & \xrightarrow{\quad} & \mathcal{C}[p(R_1)^{-1}] \xrightarrow{\simeq} KO \end{array}$$

has a unique (up to homotopy) solution $h: \mathcal{C} \rightarrow ko$ which clearly induces isomorphisms on homotopy groups. \square

Remark 2.15. It is not possible to improve these methods in the sense that one could pass from ko to KO in Thm. 1.2. In fact, in [DSS03] it is proved that there are manifolds in the kernel of the periodization map $ko_*(X) \rightarrow KO_*(X)$ which does not admit a pscm.

3. POSITIVE SCALAR CURVATURE

Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a family of smooth closed manifolds. Now we shall prove our geometric result that a \mathcal{P} -manifold carries a pscm if all P_i do. We note that neither orientability and spin structures nor regularity of the sequence P_1, P_2, \dots are needed.

Theorem 3.1. *Let $\mathcal{P} = \{P_1, P_2, \dots\}$ be a family of pscm manifolds. Then a \mathcal{P} -manifold, considered as a smooth manifold with additional structure, carries a pscm.*

As a consequence we obtain a proof of Thm. 1.3 with 2 inverted:

Corollary 3.2. *For all spaces X the kernels of $A: \Omega_*^{Spin}(X) [\frac{1}{2}] \rightarrow ko_*(X) [\frac{1}{2}]$ and $U: \Omega_*^{SO}(X) [\frac{1}{2}] \rightarrow H_*(X; \mathbb{Z} [\frac{1}{2}])$ are generated by manifolds which carry a pscm.*

Proof. As mentioned above $\ker U$ is generated by projective spaces and hypersurfaces of degree $(1, 1)$ in $\mathbb{C}P^n \times \mathbb{C}P^m$. In [GL80] it is explained why these manifolds carry a pscm: The standard Fubini-Study metric on $\mathbb{C}P^n$ is a pscm. Hypersurfaces of degree $(1, 1)$ in $\mathbb{C}P^n \times \mathbb{C}P^m$ are projective space bundles over projective spaces, and with the induced metric the fibers being totally geodesic submanifolds. Hence the O'Neill formulas [Bes86, Prop. 9.70d] guarantee pscm on total spaces.

In [KS93, Sec. 4] it is proved that $\ker A$ is generated by $\mathbb{H}P^2$ -bundles with isometric action of the structure group. One concludes [Bes86, Thm. 9.59] that there exists metrics on the total spaces such that the fibers are also totally geodesic. Since $\mathbb{H}P^2$ carries a pscm the claim again follows by the O'Neill formulas. \square

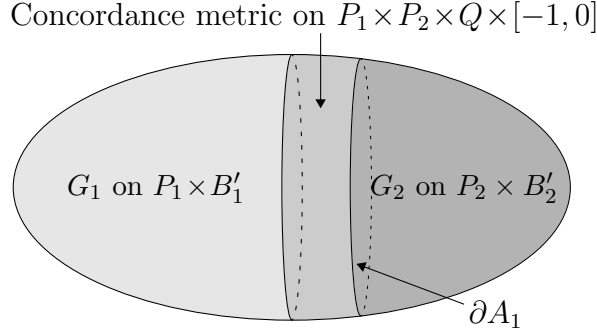
As a preliminary point, we recall that a *collar metric* on a smooth manifold M with boundary is a collar neighborhood $\partial M \times [0, 1]$ together with a metric on M which restricts to $g \times dt^2$ on $\partial M \times [0, 1]$, where g is some metric on ∂M and dt^2 is the standard metric on $[0, 1]$. Two pscm g_0 and g_1 on a closed manifold M are called

- *isotopic* if they lie in the same path component of the space of pscm on M equipped with the C^∞ topology,
- *concordant* if there exists a collar pscm H on $M \times [0, 1]$ such that $H|_{M \times \{0\}} = g_0$ and $H|_{M \times \{1\}} = g_1$.

It is well known that isotopy implies concordance.

With respect to products and scalar multiplication scalar curvature behaves as follows. If $\epsilon > 0$ and g_0, g_1 are metrics on M, N then $scal(\epsilon g_0) = \epsilon^{-1} scal(g_0)$ and $scal(g_0 \times g_1) = scal(g_0) + scal(g_1)$. Hence, if M and N are compact and N admits a pscm then $M \times N$ does.

The crucial step in the proof of Thm. 3.1 is a simple concordance argument which can be easily demonstrated in the case of a \mathcal{P} -manifold M consisting of two P_i -parts, i.e. $M = A_1 \cup A_2$. Due to the definition of a \mathcal{P} -manifold there is a submanifold $Q \subset \partial B_{12}$ such that $\phi_1(\partial A_1) = P_1 \times P_2 \times Q$ and a submanifold $B'_2 \subset B_2$ such that $\phi_2(A_2 - \mathring{A}_1) = P_2 \times B'_2$. In addition, $\partial A_1 \hookrightarrow A_2 - \mathring{A}_1$ is induced by some diffeomorphism $\psi: P_1 \times Q \rightarrow \partial B'_2$. Choose a pscm g_1 on P_1 , a metric h on Q and extend $\psi_*(g_1 \times h)$ to a collar metric h_2 on B'_2 . Now take a pscm g_2 on P_2 such that $G_2 := g_2 \times h_2$ is a pscm on $\phi_2(A_2 - A_1)$. Next extend $g_2 \times h$ to a

FIGURE 4. Construction of a pscm on $A_1 \cup A_2$

collar metric h_1 on B_1 . We find an $\epsilon > 0$ small enough such that $G_1 := (\epsilon g_1) \times h_1$ is a pscm on $\phi_1(A_1)$. Note that $\phi_1^*(G_1)$ and $\phi_2^*(G_2)$ restricted to ∂A_1 are isotopic, thus concordant. To obtain the desired pscm on M we use the concordance metric on $\partial A_1 \times [-1, 0] \subset A_1$ to join the pscm $\phi_1^*(G_1)$ restricted to $(A_1 - (\partial A_1 \times [-1, 0]))$ and the pscm $\phi_2^*(G_2)$ on A_2' (see figure 4, we set $B_1' := B_1 - (\partial B_1 \times [-1, 0])$).

The arduousness of the proof of Thm. 3.1 for \mathcal{P} -manifolds consisting of more P_i -parts merely lies in the fact that metrics on diverse submanifolds have to be chosen in a compatible way.

For simplicity we omit diffeomorphisms in the sequel. Let $M = A_1 \cup \dots \cup A_k$ be a \mathcal{P} -manifold. As above fix bicollar neighborhoods $\partial A_i \times [-1, 1]$ of $\partial A_i \subset M$, say $\partial A_i \times \{-1\} \subset A_i$, such that $(\partial A_i \times [-1, 1]) \cap A_j$ is a bicollar neighborhood for $(\partial A_i) \cap A_j$ in A_j for all j . Define a new covering by $A_1' := A_1$ and, for $i > 1$, $A_i' := A_i - (\cup_{j < i} \mathring{A}_j)$. Then one has $A_i' = P_i \times B_i'$ for appropriate $B_i' \subset B_i$.

We set $A^j = \cup_{i=j}^k A_i'$. For $1 < j \leq k$ note that A^j is modelled on $\mathbb{H}_1^n \cap \dots \cap \mathbb{H}_{j-1}^n$ and inherits collar neighborhoods $(\partial A_i \cap A^j) \times [0, 1]$ for all $i < j$. By a collar metric on A^j we understand a metric which

- extends to a smooth metric on M ,
- restricts for all $i < j$ to a product on $(\partial A_i \cap A^j) \times [0, 1]$ with the standard metric on the second factor.

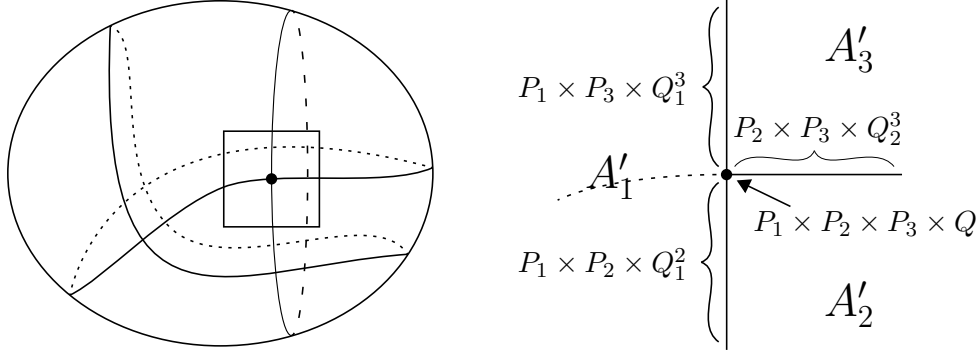
The same notation is used for other manifolds modelled on intersections of half spaces, like A_j' , B_j' and Q_I^j (defined in the sequel). All upcoming metrics are supposed to be collar metrics.

For all $1 \leq j \leq k$ and

$$(3.1) \quad I = \{i_1, \dots, i_s\}, 1 \leq i_1 < \dots < i_s \leq j-1$$

there exists a manifold Q_I^j such that

$$(3.2) \quad \left(\bigcap_{i \in I} \partial A_i \right) \cap A_j' = P_{i_1} \times \dots \times P_{i_s} \times P_j \times Q_I^j.$$

FIGURE 5. \mathcal{P} -manifold and 'deepest' point neighborhood

With this notation we have $Q_\emptyset^i = B'_i$. Note that in particular, since $(\cap_{i=1}^{k-1} \partial A_i)$ lies in A_k ,

$$\left(\bigcap_{i=1}^{k-1} \partial A_i \right) = P_1 \times \dots \times P_k \times Q_{\{1, \dots, k-1\}}^k.$$

The manifold $Q := Q_{\{1, \dots, k-1\}}^k$ can be described as the 'deepest' point of M . Now choose pscm g_i on P_i for $1 \leq i < k$ and a metric h on Q . We need the above concordance argument in the following form.

Lemma 3.3. *Assume that there is a $1 \leq j < k$ and a pscm G^{j+1} on A^{j+1} . One finds an $R \subset \partial B'_j$ such that $\partial A_j \cap A^{j+1} = P_j \times R$. In fact, $R = (\cup_{i=j+1}^k P_i \times Q_j^i)$, cf. figure 5. Assume further that $G^{j+1}|_{P_j \times R} = g_j \times h_j|_R$ for some metric h_j on B'_j . Then there exists an extension of G^{j+1} to a pscm G^j on A^j .*

Proof. We agreed that h_j is a collar metric for the induced collar neighborhood $R \times [-1, 0]$ of R in B'_j . One finds an $\epsilon > 0$ such that $(\epsilon g_j) \times h_j$ is a pscm on A'_j . Since G^{j+1} is a collar metric, $g_j \times h_j|_R$ and thus $(\epsilon g_j) \times h_j|_R$ are pscm. It is obvious that they are isotopic, hence concordant. Denote by G the concordance metric on $(P_j \times R) \times [-1, 0]$. Now we define a pscm on

$$A^j = A^{j+1} \cup (P_j \times R \times [-1, 0]) \cup (A'_j - (P_j \times R \times [-1, 0]))$$

by

$$G^{j+1} \cup G \cup \left((\epsilon g_j) \times h_j|_{B'_j - (R \times [-1, 0])} \right).$$

□

Let us describe our strategy how to construct a pscm on M in terms of a \mathcal{P} -manifold with three P_i -parts (see figure 5). Recall that we chose a metric h on Q and pscm g_1 resp. g_2 on P_1 resp. P_2 .

- (1) Extend $g_2 \times h$ to a metric h_1^3 on Q_1^3 and h_2^3 on Q_2^3 .
- (2) Extend $(g_1 \times h_1^3) \cup (g_2 \times h_2^3)$ to a metric h_3 on B'_3 .
- (3) Choose a pscm g_3 on P_3 such that $G^3 := g_3 \times h_3$ is a pscm on A'_3 .

- (4) Extend $g_3 \times h$ to a metric h_1^2 on Q_1^2 and $(g_1 \times h_1^2) \cup (g_3 \times h_2^3)$ to a metric h_2 on B_2' .
- (5) Apply Lemma 3.3 to extend G^3 to a pscm G^2 on $A_2' \cup A_3'$. Observe that Lemma 3.3 gives us a metric f_1 on $P_2 \times Q_1^2$ such that $G^2|_{P_1 \times (P_2 \times Q_1^2)} = g_1 \times f_1$.
- (6) Extend $(g_3 \times h_1^3) \cup f_1$ to a metric h_1 on B_1' .
- (7) Apply Lemma 3.3 to extend G^2 to pscm a G^1 on $A_1' \cup A_2' \cup A_3'$.

Turn back to the general case. For the upcoming construction it is very helpful to keep figure 5 in mind. Let $1 \leq j \leq k$ and $1 \leq r < j$. One verifies that

$$\partial A^j = \bigcup_{r=1}^{j-1} (\partial A_r \cap A^j)$$

and

$$\partial A_r \cap A^j = P_r \times \left(\bigcup_{i=j}^k P_i \times Q_r^i \right).$$

We shall prove the following statement by induction over j , starting with $j = k$ and ending with $j = 1$, from which Thm. 3.1 follows immediately.

Lemma 3.4. *There exists a pscm G^j on A^j with compatible product structures on ∂A^j which means that for all $1 \leq r < j$ there are metrics f_r^j on $\bigcup_{i=j}^k P_i \times Q_r^i$ such that*

$$(3.3) \quad G^j|_{\partial A_r \cap A^j} = g_r \times f_r^j,$$

where g_r are the fixed pscm from above.

Proof. For the initial step, which corresponds to the steps (1) - (3) above, one has to consider A_k' . First we define a metric h_k on B_k' . Observe that for $J \subsetneq \{1, \dots, k-1\}$

$$(3.4) \quad \partial Q_J^k = \bigcup_{\{S \subset \{1, \dots, k-1\} \mid J \subset S, |S|=|J|+1\}} P_{S-J} \times Q_S^k.$$

Step by step, starting with $|J| = k-2$ and ending with $J = \emptyset$, we extend the metric h on the 'deepest' point Q to metrics h_J^k on Q_J^k such that $h_J^k|_{(P_r \times Q_{J \cup \{r\}}^k) \subset \partial Q_J^k} = g_r \times h_{J \cup \{r\}}^k$ for $\{r\} = I - J$.

One obtains a metric $h_k := h_\emptyset^k$ on $B_k' = Q_\emptyset^k$ such that for all $1 \leq r < k$

$$(3.5) \quad h_k|_{(P_r \times Q_r^k) \subset \partial B_k'} = g_r \times h_r^k.$$

Now choose a pscm g_k on P_k such that $G^k := g_k \times h_k$ is a pscm on A_k' . Set $f_r^k := g_k \times h_r^k$. By means of 3.5 the condition 3.3 is satisfied. This finishes the induction basis.

We turn to the induction step which corresponds to the steps (4) - (5) resp. (6) - (7) above. Let $1 \leq j < k$ and G^{j+1} a pscm on A^{j+1} such that 3.3 is satisfied. First we define a metric on

$$\partial B_j' = \left(\bigcup_{r=1}^{j-1} P_r \times Q_r^j \right) \cup \left(\bigcup_{i=j+1}^k P_i \times Q_j^i \right)$$

as follows. As above one finds metrics h_r^j on Q_r^j and we consider $g_r \times h_r^j$ on $P_r \times Q_r^j$ for all $r \leq j-1$. By induction hypothesis there is the metric f_j^{j+1} on $\cup_{i=j+1}^k P_i \times Q_j^i$. Now extend

$$\left(\bigcup_{r=1}^{j-1} g_r \times h_r^j \right) \cup f_j^{j+1}$$

to a metric h_j on B'_j .

Finally we apply Lemma 3.3 to obtain a pscm G^j on A^j . We note that the concordance metric in Lemma 3.3 does not alter the g_r factor on $P_r \times P_j \times Q_r^j$. Hence, for all $r \leq j-1$, there is an induced metric f_r on $P_j \times Q_r^j$ such that $G^j|_{P_r \times (P_j \times Q_r^j)} = g_r \times f_r$. By means of the induction hypothesis one verifies that condition 3.3 is satisfied with $f_r^j := f_r \cup f_r^{j+1}$ on $\cup_{i=j}^k P_i \times Q_r^i$. \square

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